Exercise session 1

Definition 1. Let f, g be functions $\mathbb{N} \to \mathbb{R}$.

$$f \in \mathcal{O}(g) \quad \text{means} \quad (\exists c, n_0 \in \mathbb{N})(\forall n \geq n_0)(|f(n)| \leq c|g(n)|) \qquad (\leq)$$

$$f \in o(g) \quad \text{means} \quad (\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)(|f(n)| < \varepsilon|g(n)|) \qquad (<)$$

$$\text{or equivalently} \quad \lim_{n\to\infty} \frac{|f(n)|}{|g(n)|} = 0$$

$$f \in \Omega(g) \quad \text{means} \quad g \in \mathcal{O}(f) \qquad (\geq)$$

$$f \in \omega(g) \quad \text{means} \quad g \in o(f) \qquad (>)$$

$$f \in \Theta(g) \quad \text{means} \quad f \in \mathcal{O}(g) \cap \Omega(g) \qquad (=)$$

Remark 1. It is very common to write $f = \mathcal{O}(g)$ instead of $f \in \mathcal{O}(g)$. Note however that it is not true that $f = \mathcal{O}(g)$ and $h = \mathcal{O}(g)$ implies $f = h$.

Exercise 1.

1. Let $f(n) = pn^3 + qn^2 + rn + s$ for some $p, q, r, s \in \mathbb{R}$.
   Show $f(n) \in \mathcal{O}(n^3)$ and $f(n) \in o(n^4)$.
2. Show $\sin(n) \in \mathcal{O}(1)$ and $\sin(n) \notin o(1)$.
3. Show that $\mathcal{O}(f + g) = \mathcal{O}(\max(f, g))$ for non-negative functions f, g (i.e. $f(n) \geq 0$ and $g(n) \geq 0$ for all n), where $f + g$ is the function $x \mapsto f(x) + g(x)$ and $\max(f, g)$ is the function $x \mapsto \max(f(x), g(x))$.
4. Show $n^{\log n} \in \mathcal{O}(2^n)$.

Solution.

1. (a) Let $c = |p| + |q| + |r| + |s|$ and $n_0 = 1$, then for $n \geq n_0$ we have $|pn^3 + qn^2 + rn + s| \leq |p|n^3 + |q|n^2 + |r|n + |s| \leq c|n^3|$.

1. (b) We have
$\lim_{n\to\infty} |pn^3 + qn^2 + rn + s|/n^4 \leq \lim_{n\to\infty}(|p|n^3 + |q|n^2 + |r|n + |s|)/n^4 = \lim_{n\to\infty} |p|/n + |q|/n^2 + |r|/n^3 + |s|/n^4 = 0$.

2. (a) Since $0 \leq |\sin(n)| \leq 1$, taking $c = 1$ and $n_0 = 0$ works.
2. (b) The limit $\lim_{n\to\infty} |\sin(n)|$ does not converge.

3. ($\subseteq$) If $h \in \mathcal{O}(f + g)$ then $\exists c, n_0\ \forall n > n_0 :\ |h(n)| \leq c|(f + g)(n)| = c|f(n) + g(n)| \leq 2c|\max(f(n), g(n))|$. So by setting $c' = 2c$ and $n_0' = n_0$ we have $|h(n)| \leq c'|\max(f(n), g(n))|$ for $n \geq n_0'$ so $h \in \mathcal{O}(\max(f, g))$.
3. ($\supseteq$) If $h \in \mathcal{O}(\max(f, g))$ then $\exists c, n_0\ \forall n > n_0 :\ |h(n)| \leq c|\max(f, g)(n)| \leq c|(f + g)(n)|$. So with the same c and n_0 we have $h \in \mathcal{O}(f + g)$.

4. First note that $(\log n)^2 \leq n$ when $n \geq 16$. Since 2^n is an increasing function (i.e. $2^a > 2^b$ if $a > b$) we have $n^{\log n} = 2^{(\log n)^2} \leq 2^n$ for $n \geq 16$. So choose $c = 1$ and $n_0 = 16$.
You are allowed to assume that when $n \to \infty$, n grows faster than any power of $\log n$.
To see why $(\log n)^2 \leq n$ for $n \geq 16$, note that for $n = 16$ we have $(\log n)^2 = n$, and the derivative of $(\log n)^2$ is always smaller than the derivative of n for $n \geq 16$. So n grows faster and we have the required inequality.