Exercise session 1

Definition 1. Let f, g be functions $\mathbb{N} \to \mathbb{R}_{\geq 0}$.

$$f \in \mathcal{O}(g) \quad \text{means} \quad (\exists c, n_0 \in \mathbb{N})(\forall n \geq n_0)(f(n) \leq cg(n)) \qquad (\leq)$$

$$f \in o(g) \quad \text{means} \quad (\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)(f(n) < \varepsilon g(n)) \qquad (<)$$

$$\text{or equivalently} \quad \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

$$f \in \Omega(g) \quad \text{means} \quad g \in \mathcal{O}(f) \qquad (\geq)$$

$$f \in \omega(g) \quad \text{means} \quad g \in o(f) \qquad (>)$$

$$f \in \theta(g) \quad \text{means} \quad f \in \mathcal{O}(g) \cap \Omega(g) \qquad (=)$$

Remark 1. It is very common to write $f = \mathcal{O}(g)$ instead of $f \in \mathcal{O}(g)$. Note however that it is not true that $f = \mathcal{O}(g)$ and $h = \mathcal{O}(g)$ implies $f = h$.

Exercise 1.

1. Let $f(n) = pn^3 + qn^2 + rn + s$ for some $p, q, r, s \in \mathbb{R}$.
   Show $f(n) \in \mathcal{O}(n^3)$ and $f(n) \in o(n^4)$.
2. Show $|\sin n| \in \mathcal{O}(1)$ and $|\sin n| \notin o(1)$.
3. Show $\mathcal{O}(f + g) = \mathcal{O}(\max(f, g))$, where $\max(f, g)$ is the function $x \mapsto \max(f(x), g(x))$.
4. Show $n^{\log n} \in \mathcal{O}(2^n)$.

Solution. 1. (a) Let $c = p + q + r + s$ and $n_0 = 1$, then for $n \geq n_0$ we have $pn^3 + qn^2 + rn + s \leq cn^3$.

1. (b) We have $\lim_{n\to\infty}(pn^3 + qn^2 + rn + s)/n^4 = \lim_{n\to\infty} p/n + q/n^2 + r/n^3 + s/n^4 = 0$.

2. (a) Since $0 \leq |\sin(n)| \leq 1$, taking $c = 1$ and $n_0 = 0$ works.

2. (b) The limit $\lim_{n\to\infty} |\sin(n)|$ does not converge.

3. ($\subseteq$) If $h \in \mathcal{O}(f + g)$ then $\exists c, n_0\ \forall n > n_0 :\ h(n) \leq c(f + g)(n) = c(f(n) + g(n)) \leq 2c \max(f(n), g(n))$.

($\supseteq$) If $h \in \mathcal{O}(\max(f, g))$ then $\exists c, n_0\ \forall n > n_0 :\ h(n) \leq c \max(f, g)(n) \leq c(f + g)(n)$.

4. Hint: first show that $(\log n)^2 \leq n$ when $n \geq 1$, and then use that 2^n is increasing.