Complexity Theory

Werkcollege 1 - Exercises to solve in class

2 April, 2014

Exercise 1. The MIT museum contains a kinetic sculpture by Arthur Ganson called *Machine with Concrete*. It consists of 13 gears connected to one another in a series such that each gear moves 50 times slower than the previous one. The fastest gear is constantly rotated by an engine of 212 rotations per minute, and has been rotating in the same way non-stop for many years. The slowest gear is fixed to a block of concrete and thus cannot move at all. Explain why this machine does not break apart.

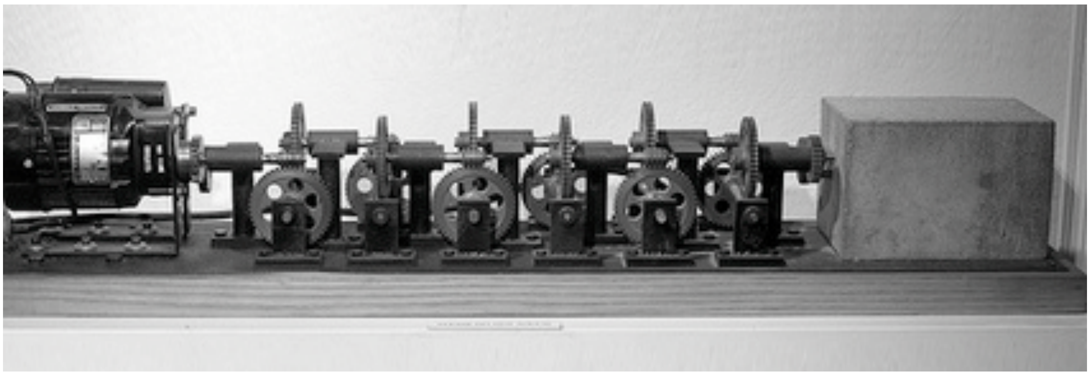

Figure 1: *Machine with concrete*, by Arthur Ganson.

Exercise 2. Prove or disprove the following:

1. If $f(n) = n^k$ and $g(n) = c^n$, for any constants $k > 0$ and $c > 1$, then $f(n) = o(g(n))$.
2. If $f(n) = \Theta(g(n))$, then $2^{f(n)} = \Theta(2^{g(n)})$.
3. $f(n) = 2^{O(\log n)}$ if and only if $f(n) = O(n^c)$ for some constant c.
4. (Bonus) $f(n) = 2^{\Theta(\log n)}$ if and only if $f(n) = \Theta(n^c)$ for some c.

Exercise 3. Consider the following algorithm for multiplication of natural numbers:

```
SLOW-MULTIPLY(a, b)
t ← 0
while b > 0
    do t ← t + a
       b ← b − 1
return t
```

(a) Supposing that you can do summation in a single step, and so each line of code takes one time step, how many steps does it take for this algorithm to multiply two natural numbers, a having n digits, and b having m digits? (answer using big-O notation, disregarding additive and multiplicative constants)

Now consider the following algorithm instead, where $\text{DIGIT}(b, i)$ can be used to extract the i-th decimal digit of b (e.g., if $b = 56789$, then $\text{DIGIT}(b, 0) = 9, \text{DIGIT}(b, 1) = 8, \text{DIGIT}(b, 2) = 7$, *etc*):

```
SCHOOL-MULTIPLY(a, b)
t ← 0
i ← 0
while 10^i ≤ b
    do t ← t + 10^i × (a × DIGIT(b, i))
       i ← i + 1
return t
```

(b) Assuming once again that each line takes one time step, how many steps does it take for this algorithm to multiply two natural numbers, a having n digits, and b having m digits?

(c) We can us the SLOW-MULTIPLY algorithm to implement the operation $a \times \text{DIGIT}(b, i)$ (multiplication with a digit) efficiently using summation only (it will take a constant number of steps, since $\text{DIGIT}(b, i)$ is at most 9). How can we implement $10^i \times x$ (multiplication with a power), in $O(i)$ many steps, so that the SCHOOL-MULTIPLY algorithm turns out to only need summation to be efficient? What is the complexity of the algorithm when done in this way?

Exercise 4. Prove the following two statements:

1. The set $A = \{\langle x, y\rangle | y = 2^x\}$ is in P; and
2. The function $f(x) = 2^x$ is not computable in polynomial-time.

Why aren't they contradictory?

Exercise 5. Show that if A and B are two sets in NP, then so are $A \cup B$ and $A \cap B$.